

# Relativistic Kinematics of Two-Parametric Riemann Surface in Genus Two

A.V. Nazarenko

Bogolyubov Institute for Theoretical Physics,  
14-b, Metrologichna Str., Kiev 03680, Ukraine  
e-mail: nazarenko@bitp.kiev.ua

Yu.A. Kulinich

Astronomical Observatory, Ivan Franko National University of Lviv,  
8, Kyryla and Methodia Str., Lviv 79005, Ukraine  
e-mail: kul@astro.franko.lviv.ua

## Abstract

It is considered a model of compact Riemann surface in genus two, represented geometrically by two-parametric hyperbolic octagon with an order four automorphism and described algebraically by the corresponding Fuchsian group. Introducing the Fenchel–Nielsen variables, we compute the Weil–Petersson (WP) symplectic two-form for parameter space and analyze the closed isoperimetric orbits of octagons. WP-Area in parameter space and the canonical action–angle variables for the orbits are found. Exploiting the ideas from the loop quantum gravity, we generate relativistic kinematics by the Lorentz boost and quantize WP-area. We treat the evolution in terms of global variables within the “big bounce” concept.

**Keywords:** Riemann surfaces in genus two; geometrodynamics; area quantization

# 1 Introduction

The Riemann surfaces are often used in the problems of string theory [1], lower-dimensional gravity [2, 3] and quantum geometry [4]. Sometimes, chaotic behavior in nature can be also related with non-trivial geometry [5, 6].

Here we study free geometrodynamics or kinematics of the surface in genus two associated with hyperbolic octagon which is embedded into Poincaré disk and stable under rotation by  $\pi/2$ . Identifying the opposite sides of such a domain, it is enough to use two real parameters in order to describe the octagon geometry and the form of Fuchsian group generators. To generate a geometry evolution, we operate by parameter space invariants instead of Riemannian metric which is unknown for us. Note that the involution of similar surfaces and the associated generators were discussed in [7, 8].

Defining the Teichmüller space [9] for a family of the given surfaces, we find the Fenchel–Nielsen variables regarding as global coordinates on it and permitting us to endow the parameter space with the Weil–Petersson (WP) symplectic two-form due to the Wolpert’s theorem [10]. Further, we demonstrate that the parameter space can be densely covered by the closed curves of the constant octagon perimeter and describe the diffeomorphism produced by isoperimetric constraint. We consider the set of isoperimetric orbits as a tool for further parametrization and quantization [11].

We determine the canonically conjugate action–angle variables for isoperimetric orbits by identifying the action with WP-area of domain bounded by the orbit in parameter space. This is a main point of our approach using the integral characteristics in a contrast with the formalism based on local Fenchel–Nielsen parameters of the surface [4]. In this way, we touch the problem of WP-area quantization, which is similar to the one of the loop quantum gravity/cosmology [12].

In order to describe geometrodynamics and to perform area quantization, we extend the algebra of action–angle variables up to generators of  $\mathfrak{su}(1,1)$  associated with the Lorentz algebra in (2+1)-dimensional space-time. Such an approach permits us to formulate the relativistic dynamics of Riemann surface as a canonical transformation

generated by the boost. It is realized in Section 4 and leads to “big bounce” scenario [13, 14] in parameter space. We note that the quantization of the universe represented by the Riemann surface in genus two was formulated algebraically in [3].

## 2 Model Octagon and Riemann Surface

The Poincaré model of two-dimensional hyperbolic space is given by open disk  $\mathbb{D} = \{z = x + iy \mid |z| < 1\}$  and the metric  $ds^2 = 4|dz|^2/(1 - |z|^2)^2$ . The geodesic in  $(\mathbb{D}, ds^2)$  is an circle arc inside  $\mathbb{D}$  with radius  $R$  and center at the point  $z_0 = \sqrt{1 + R^2} \exp(i\phi)$  lying beyond the unit disk. In particular case, the geodesics emanating from the origin are the Euclidean straight lines (diameters). All geodesics intersect the boundary  $\partial\mathbb{D}$  orthogonally.

The group of all orientation-preserving isometries  $\gamma$  of  $(\mathbb{D}, ds^2)$ , denoted by  $\text{Isom}^+(\mathbb{D})$ , acts via the Möbius transformation:

$$z \mapsto \gamma[z] = \frac{uz + v}{\bar{v}z + \bar{u}}, \quad z \in \mathbb{D}, \quad (1)$$

where  $u$  and  $v$  satisfy relation  $|u|^2 - |v|^2 = 1$ ;  $\bar{u}, \bar{v}$  are the complex conjugates. Thus, it is convenient to identify a generator  $\gamma$  with an element of group

$$\text{SU}(1, 1) = \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \mid |u|^2 - |v|^2 = 1 \right\}. \quad (2)$$

The Riemann surface  $S$  is understood here as a compact two-dimensional orientable manifold with the metric of constant negative curvature. Such a surface is obtained from hyperbolic simply connected octagon  $\mathcal{F}$  in  $\mathbb{D}$  via gluing opposite sides formed by eight geodesic arcs, whose intersections serve as vertices.

Let the vertices be at the points  $a \exp(ik\pi/2), b \exp[i(\alpha + k\pi/2)]$  (Fig. 1, left panel), where  $0 < \alpha < \pi/2$ ,  $0 < a, b < 1$  and  $k = \overline{0, 3}$ . We also require that the sum of inner angles of  $\mathcal{F}$  equals to  $2\pi$  and hyperbolic  $\text{Area}(\mathcal{F}) = 4\pi$  in the case of the surface in genus two.

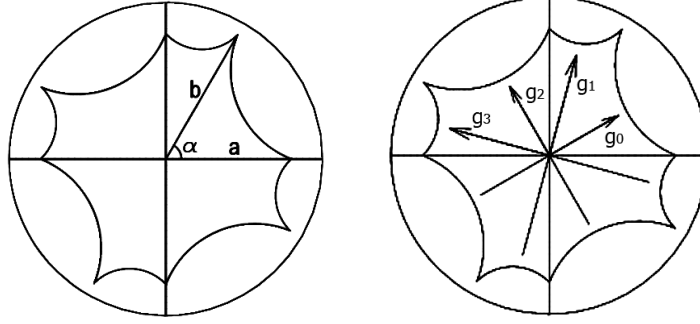


Figure 1: Octagon with  $a = 0.8$ ,  $\alpha = \pi/3$  and generators  $g_k$  of Fuchsian group.

Choosing two parameters  $(a, \alpha)$  as independent real variables, we find that  $b$  and the inner angle  $\beta$  by vertices  $a \exp(ik\pi/2)$  are

$$b = \left( \sqrt{2}a \cos \tilde{\alpha} \right)^{-1}, \quad \tan \beta = \frac{1 - a^2}{1 - b^2}; \quad \tilde{\alpha} = \alpha - \frac{\pi}{4}. \quad (3)$$

These allow us to determine the region  $\mathcal{A}$  of variety of parameters  $(a, \alpha)$ :

$$-\pi/4 < \tilde{\alpha} < \pi/4, \quad \left( \sqrt{2} \cos \tilde{\alpha} \right)^{-1} < a < 1, \quad (4)$$

which is shown in Fig. 2 below.

To obtain the regular hyperbolic octagon, we should put  $a = 2^{-1/4}$ ,  $\alpha = \pi/4$ .

In the case at hand, the octagon boundary  $\partial\mathcal{F}$  is formed by geodesics of two kinds (labeled by “ $\pm$ ” below), which are completely determined by the radii  $R_{\pm}$  and the angles  $\phi_{\pm} + k\pi/2$ ,  $k = \overline{0, 3}$ , defining the positions of the circle (arc) centers. Geometrical conditions result in parametrization:

$$R_{\pm} = \frac{1}{2a} \sqrt{T_{\pm}^2 + (1 - a^2)^2}, \quad \phi_{\pm} = \arctan \left[ \left( \frac{T_{\pm}}{1 + a^2} \right)^{\pm 1} \right], \quad (5)$$

here  $T_{\pm} = a^2 \pm \tan \tilde{\alpha}$  and  $0 < \phi_+ < \alpha < \phi_- < \pi/2$ .

Here, we connect the model octagon  $\mathcal{F}$  with the corresponding Riemann surface  $S$  and the Fuchsian group  $\Gamma \subset \text{Isom}^+(\mathbb{D})$  isomorphic to fundamental group  $\pi_1(S)$ .

Since the opposite sides of  $\mathcal{F}$  have the same lengths, we can define isometry  $g_k \in \text{Isom}^+(\mathbb{D})$  mapping geodesic boundary segment  $s_{k+4}$  onto  $s_k$  for all  $k = \overline{0, 3}$  (see Fig. 1, right panel). Identifying any  $z \in s_{k+4}$  with  $g_k[z] \in s_k$ , we obtain a closed surface. Four isometries  $g_k$  and their inverses  $g_k^{-1}$  generate Fuchsian group  $\Gamma$  with a single relation:

$$g_0 g_1^{-1} g_2 g_3^{-1} g_0^{-1} g_1 g_2^{-1} g_3 = \text{id}, \quad (6)$$

and we then define surface  $S$  as a quotient  $\mathbb{D}/\Gamma$ .

Our calculations give the dependence of  $g_0$  and  $g_1$  on  $(a, \alpha)$ :

$$g_0 = N(a, \tilde{\alpha}) \begin{pmatrix} a(1 - \tan \tilde{\alpha}) & (a^2 - \tan \tilde{\alpha}) + i(1 - a^2) \\ (a^2 - \tan \tilde{\alpha}) - i(1 - a^2) & a(1 - \tan \tilde{\alpha}) \end{pmatrix}, \quad (7)$$

$$g_1 = N(a, \tilde{\alpha}) \begin{pmatrix} a(1 + \tan \tilde{\alpha}) & (1 - a^2) + i(a^2 + \tan \tilde{\alpha}) \\ (1 - a^2) - i(a^2 + \tan \tilde{\alpha}) & a(1 + \tan \tilde{\alpha}) \end{pmatrix}, \quad (8)$$

here

$$N(a, \tilde{\alpha}) = \frac{-\cos \tilde{\alpha}}{\sqrt{(1 - a^2)(2a^2 \cos^2 \tilde{\alpha} - 1)}}.$$

The remaining generators are simply obtained by rotations:

$$g_{2,3} = R_{\frac{\pi}{2}} g_{0,1} R_{\frac{\pi}{2}}^{-1}, \quad g_k^{-1} = R_{\pi} g_k R_{\pi}^{-1}; \quad (9)$$

$$R_{\varphi} = \begin{pmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{pmatrix}.$$

Let us mark a surface by generators of  $\Gamma$ . Two marked surfaces  $(S, \Gamma)$  and  $(S', \Gamma')$  are called marking equivalent if there exists an isometry  $\gamma : S \rightarrow S'$  satisfying  $g'_k = \gamma g_k \gamma^{-1}$ . Then all marking equivalent surfaces form a marking equivalence class  $[S, \Gamma]$  representing the Riemann surface  $S$ .

The set of all marking equivalence classes of the closed and compact Riemann surfaces in genus  $g$  forms the Teichmüller space  $\mathcal{T}_g$ . The real dimension of  $\mathcal{T}_g$  like vector space equals to  $6g - 6$  in accordance with the Riemann–Roch theorem. In our case, the Riemann surfaces result in the subset of total  $\mathcal{T}_2$ .

### 3 Structure of Parameter Space $\mathcal{A}$

#### 3.1 Symplectic Two-Form

A hyperbolic Riemann surface of genus  $g$  without boundary always contains a system of  $3g - 3$  simple closed geodesics that are neither homotopic to each other nor homotopically trivial. The cut along these geodesics always decomposes surface into  $2g - 2$  pairs of pants (three-holed spheres), playing a role of natural building blocks for Riemann surface [15].

In the case at hand, surface  $S$  is two-holed torus which can be decomposed into two pairs of pants by a system of three closed geodesics. This surgery results in computing the Fenchel–Nielsen (FN) parameters: lengths  $\ell_k$  of these geodesics and the corresponding twists  $\tau_k$  (see [9, 15]).

For one of possible pants decompositions, we have found these quantities [16]:

$$\ell_{1,2} = 2 \operatorname{arccosh} \frac{a^2}{1 - a^2}, \quad \ell_3 = 2 \ln \frac{1 + a}{1 - a}, \quad (10)$$

$$\tau_{1,2} = \operatorname{arccosh} \left[ \frac{2a^2 - 1}{a^2(1 - b^2)} - 1 \right], \quad \tau_3 = \ln \frac{1 + a}{1 - a}. \quad (11)$$

Since the Teichmüller space  $\mathcal{T}_2$  is homeomorphic to  $\mathbb{R}^6$ , one can identify the FN variables with global coordinates on it. Moreover, the Teichmüller space carries additional structure, namely, the Weil–Petersson (WP) symplectic two-form. Due to a theorem by Wolpert [9, 17], WP-form for compact closed Riemann surfaces of genus  $g$  takes on a particularly simple form in terms of FN variables,

$$\omega_{\text{WP}} = \frac{1}{2} \sum_{k=1}^{3g-3} d\ell_k \wedge d\tau_k, \quad (12)$$

which is invariant with respect to any pants decomposition. Introducing  $\theta_k = 2\pi\tau_k/\ell_k$ , the simple Dehn twist  $\theta_k \rightarrow \theta_k + 2\pi$  gives us isometrically the same surface.

Substituting the expressions for  $\ell_k$  and  $\tau_k$  into (12), WP-form becomes

$$\omega_{\text{WP}} = \frac{8a}{(1 - a^2)(2a^2 \cos^2 \tilde{\alpha} - 1)} da \wedge d\tilde{\alpha}. \quad (13)$$

To verify the uniqueness of last formula, we can consider another pants decomposition. In terms of our parameters, a new decomposition simply leads to replacement,

$$a \leftrightarrow b, \quad \tilde{\alpha} \leftrightarrow -\tilde{\alpha}, \quad (14)$$

in length and twist functions of previous decomposition. Although the set of new functions is obtained, the resulting two-form  $\omega_{\text{WP}}$  remains the same.

Thus, domain  $\mathcal{A}$  of admissible parameters  $(a, \alpha)$  is symplectic manifold  $(\mathcal{A}, \omega_{\text{WP}})$ . We may also treat two-form (13) as an area element of  $\mathcal{A}$ .

## 3.2 Isoperimetric Orbits

Since the hyperbolic area of admissible octagons is always equal to  $4\pi$ , a simplest way to control the surface changes globally consists in consideration of octagon perimeter:

$$P = 8 \operatorname{arccosh} \frac{1 - a^2 b^2 + \sqrt{(1 - a^2)^2 + (1 - b^2)^2}}{(1 - a^2)(1 - b^2)}. \quad (15)$$

Although perimeter  $P$  is obviously invariant under automorphism and pants decomposition, it can also take on the same value for different pairs  $(a, \alpha)$ . We are aiming to describe the corresponding orbits. On the contrary to our case, two parameters of *flat* octagon with the same automorphism and the homothety property are exactly defined by fixing its area and perimeter.

For further calculations it is useful to introduce two auxiliary quantities:

$$T \equiv \tanh(P/16), \quad \varepsilon = \pm 1, \quad (16)$$

where the latter one reflects an existence of two sheets in  $\mathcal{A}$  labeled by sign  $\tilde{\alpha}$ .

For a given  $T(P)$ , maximal and minimal values of  $a$  at  $\tilde{\alpha} = 0$  are

$$a_{\pm}(T) = \frac{1}{2} \sqrt{2 + T^2 \pm \sqrt{(2 + T^2)^2 - 8}}. \quad (17)$$

Solution to equation  $(2 + T^2)^2 = 8$  is  $T_{\text{reg}} = \sqrt{2\sqrt{2} - 2}$  that results in quantities  $P_{\text{reg}} = 8 \operatorname{arccosh}(5 + 4\sqrt{2})$  and  $a_{\text{reg}} = 2^{-1/4}$  corresponding to

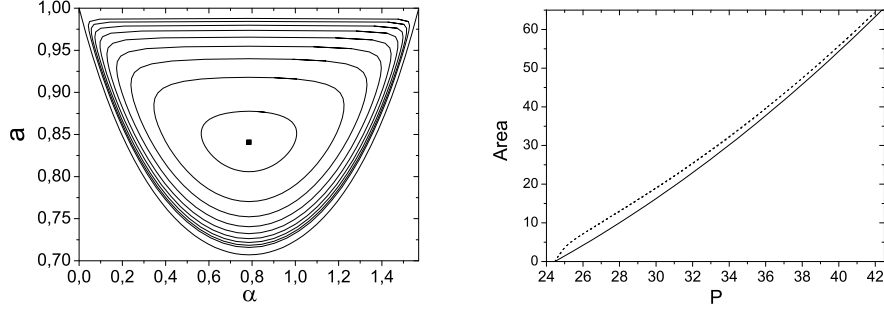


Figure 2: Left panel: Parameter space  $\mathcal{A}$  and orbits of constant perimeter. Right panel: WP-Area of domain bounded by isoperimetric curve. Solid curve is computed numerically; dashed curve is given by analytic approximation (43).

the regular octagon. Thus,  $P_{\text{reg}} \approx 24.457$  is a minimal value of  $P$  among possible ones and trajectory in  $\mathcal{A}$  for  $P_{\text{reg}}$  is contracted to a point.

Isoperimetric orbits can then be parametrized as

$$a(T, \varphi) = \frac{1}{2} \sqrt{2 + T^2 + \cos \varphi \sqrt{(2 + T^2)^2 - 8}}, \quad (18)$$

$$\tilde{\alpha}(T, \varphi) = \arctan \frac{\sqrt{2} \sqrt{(2 + T^2)^2 - 8} \sin \varphi}{2 \sqrt{3T^2 - 2 - \cos \varphi \sqrt{(2 + T^2)^2 - 8}}}, \quad (19)$$

where cyclic variable  $\varphi \in [0, 2\pi)$  is used.

These orbits are shown in Fig. 2 (left panel) for  $P$  from  $P = 25$  to  $P = 41$  with step 2. Although a role of octagon perimeter is unclear from physical point of view, it is important that the isoperimetric constraint guarantees the dense covering of  $\mathcal{A}$ , which has a unique shape, by the corresponding orbits. This fact allows us to quantize  $(\mathcal{A}, \omega_{\text{WP}})$  in a spirit of [11]. To realize it, we should identify the Weil-Petersson area  $A_{\text{WP}}(P)$  of domain in  $\mathcal{A}$ , bounded by curve for fixed  $P$ , with an action variable, that is, integral of “motion”. Then, quantization has to give us the number of quantum “cells” inside of the domain. It seems enough if additional physics (except quantization) is not used. However, we shall require  $\text{SO}(2,1)$  symmetry of dynamics in accordance with Lorentzian (2+1)-dimensional gravity.



Using the symplectic form (13), let us define WP-area:

$$\begin{aligned} A_{\text{WP}}(P) &\equiv \int_{P=\text{const}} \omega_{\text{WP}} \\ &= \int_{x_-(T)}^{x_+(T)} \frac{8dx}{(1-x)\sqrt{2x-1}} \operatorname{arccosh} f(x, T), \end{aligned} \quad (20)$$

where  $x \equiv a^2$ ; functions  $x_{\pm}(T) \equiv a_{\pm}^2(T)$  are determined by (17);

$$f(x, T) = \frac{1}{\sqrt{1-T^2}} \sqrt{\frac{2x-1}{x}} \sqrt{\frac{T^2-x}{T^2-2x+1}}, \quad (21)$$

and convention (16) is applied.

Analytic estimation of WP-area is made in Appendix A. We show in Fig. 2 (right panel) that  $A_{\text{WP}}^{(0)}(P)$  tends to numerically computed  $A_{\text{WP}}(P)$  at large  $P$ , but behavior at  $P \rightarrow P_{\text{reg}}$  looks incorrect. However, we shall see that the derivative  $dA_{\text{WP}}(P)/dP$  can be calculated explicitly.

### 3.3 Global Canonical Variables

Finding the canonical variables for isoperimetric orbits, let us define

$$Q(x, T) = \frac{\sqrt{2}}{4} \frac{1-T^2}{\sqrt{T^2-x_-}} \left[ \frac{F(u, k)}{1-T^2} - \frac{\Pi(u, \nu_1, k)}{1-x_-} + \frac{\Pi(u, \nu_2, k)}{T^2-2x_-+1} \right], \quad (22)$$

where  $F$  and  $\Pi$  are elliptic integrals of the first and third kind, respectively.

Amplitude  $u$ , module  $k$ , and parameters  $\nu_1$  and  $\nu_2$  are given by

$$\begin{aligned} u &= \sqrt{\frac{x-x_-}{x_+-x_-}}, & k &= \sqrt{\frac{x_+-x_-}{T^2-x_-}}, \\ \nu_1 &= \frac{x_+-x_-}{1-x_-}, & \nu_2 &= 2 \frac{x_+-x_-}{T^2-2x_-+1}. \end{aligned} \quad (23)$$

Then, symplectic WP-form becomes

$$\omega_{\text{WP}} = \varepsilon dQ(x, T) \wedge dP. \quad (24)$$

Integrating over  $x$ , we see that

$$\begin{aligned}\frac{dA_{\text{WP}}}{dP} &\equiv \oint_{P=\text{const}} \varepsilon dQ(x, T) = \int_{x_-}^{x_+} dQ(x, T) - \int_{x_+}^{x_-} dQ(x, T) \\ &= 2Q(x_+, T),\end{aligned}\tag{25}$$

where amplitude  $u = 1$  results in the complete elliptic integrals.

Defining an action variable (or “angular momentum”) as

$$J(P) = \frac{1}{4\pi} A_{\text{WP}}(P),\tag{26}$$

we find the angle variable  $\Phi$  from equation  $\omega_{\text{WP}} = 2d\Phi \wedge dJ$ ; the Poisson bracket is then  $\{J, \Phi\}_{\text{WP}} = 1$ . One first has that

$$d\Phi = \frac{\pi\varepsilon}{Q(x_+, T)} dQ(x, T), \quad \oint_{P=\text{const}} d\Phi = 2\pi,\tag{27}$$

where the rule of calculation of integrals containing  $\varepsilon$  (see (25)) is applied.

At fixed  $P$ , we come to expression for the angle variable:

$$\Phi = \pi\varepsilon \frac{Q(x, T)}{Q(x_+, T)}, \quad \Phi \in [-\pi, \pi].\tag{28}$$

Now it seems trivially to quantize the system in terms of  $J$  and  $\Phi$  what leads immediately to estimation (in the Planck units) for relatively large  $n$ :

$$A_{\text{WP}} \sim 4\pi n, \quad n \in \mathbb{N}.\tag{29}$$

We specify this formula below due to consideration of the relativity theory.

## 4 Relativistic Kinematics

At this stage, geometrodynamics of Riemann surface within the considered model is ambiguous because of purely gauge nature. There are only the geometric constraints defining the “physical sector” of parameters variety and no definitions of time and the Hamiltonian function having the physical meaning and generating an evolution of  $J$  and  $\Phi$ . However, we appeal here

to (2+1)-dimensional gravity where  $\text{SO}(2, 1) \sim \text{SU}(1, 1)$  plays a role of the Lorentz group. It allows us to construct a dynamical model with the same symmetry as follows.

Combining  $J$  and  $\Phi$ , we extend the set of observables up to

$$J_0 = J, \quad J_{\pm} = \sqrt{J^2 - C} \exp(\mp i\Phi), \quad (30)$$

where  $C$  is a constant such that  $J^2 \geq C \geq 0$ .

The Poisson algebra of new variables is  $\mathfrak{su}(1, 1)$  Lie algebra:

$$\{J_+, J_-\}_{\text{WP}} = 2iJ_0, \quad \{J_{\pm}, J_0\}_{\text{WP}} = \pm iJ_{\pm}. \quad (31)$$

More generally, generators  $J_{0,\pm}$  may be replaced by an infinite number of quantities  $L_n = J \exp(in\Phi)$ ,  $n \in \mathbb{Z}$ , generating the Witt algebra.

Since the evolution of conservative system is usually described by canonical transformations, we find that indeed  $\text{SU}(1, 1)$  transformations are canonical transformations of a given system. Let us introduce the matrix

$$\mathcal{M} = \begin{pmatrix} J_0 & J_+ \\ J_- & J_0 \end{pmatrix}, \quad (32)$$

whose determinant  $C \equiv J_0^2 - J_+ J_-$  is the Casimir of  $\mathfrak{su}(1, 1)$  algebra.

Action of matrix  $U \in \text{SU}(1, 1)$  looks like  $\mathcal{M} \mapsto \tilde{\mathcal{M}} = U\mathcal{M}U^\dagger$  and preserves the Casimir,  $\det \tilde{\mathcal{M}} = C$ .

Relativistic kinematics of the surface can be generated by the boost:

$$U_\tau = \begin{pmatrix} \cosh(\tau/2) & \sinh(\tau/2) \\ \sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix}. \quad (33)$$

Computing  $\mathcal{M}(\tau) = U_\tau \mathcal{M}(0) U_\tau^\dagger$ , one derives the trajectories for  $J_{0,\pm}$  or, equivalently, for  $J$  and  $\Phi$ :

$$J(\tau) = \check{J} \cosh \tau + \sqrt{\check{J}^2 - C} \sinh \tau \cos \check{\Phi}, \quad (34)$$

$$\Phi(\tau) = \arccos \frac{\check{J} \sinh \tau + \sqrt{\check{J}^2 - C} \cosh \tau \cos \check{\Phi}}{\sqrt{J^2(\tau) - C}}, \quad (35)$$

where  $\check{J} = J[P(a_0, \alpha_0)]$ ,  $\check{\Phi} = \Phi[a_0, P(a_0, \alpha_0)]$  are the values of functions (26), (28) at  $\tau = 0$ .

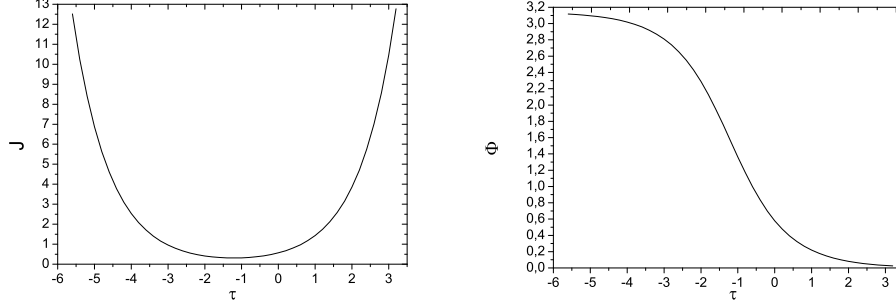


Figure 3: Angular momentum  $J$  and angle  $\Phi$  as functions of internal time  $\tau$ ;  $C = 0$  and initial data are  $a_0 = 0.8$ ,  $\alpha_0 = \pi/3$ . The surface starts with an infinite WP-area at  $\tau \rightarrow -\infty$ , contracts and bounces to expand again to infinite area at  $\tau \rightarrow \infty$ .

On the other hand, the evolution can be described by the Hamiltonian equations:

$$\partial_\tau J \equiv \{J, H\}_{\text{WP}} = \sqrt{J^2 - C} \cos \Phi, \quad (36)$$

$$\partial_\tau \Phi \equiv \{\Phi, H\}_{\text{WP}} = -\frac{J}{\sqrt{J^2 - C}} \sin \Phi, \quad (37)$$

where the Hamiltonian

$$H(J, \Phi, C) = \sqrt{J^2 - C} \sin \Phi \quad (38)$$

belongs to  $\mathfrak{so}(2, 1)$  algebra.

Of course, the form of the Hamiltonian depends generally on the physical problem under consideration. In relativistic cosmology, different scenarios lead to modifications of (38) which are discussed, for instance, in [14].

In our case, time dependence of functions  $J(\tau)$ ,  $\Phi(\tau)$  is sketched in Fig. 3. The figures demonstrate a “bounce” in parameter space. Finding the zeroes of derivatives of (34), (35) with respect to  $\tau$ , the bounce characteristics are

$$\tau_b = -\text{arctanh} \left( \frac{\sqrt{\check{J}^2 - C}}{\check{J}} \cos \check{\Phi} \right),$$

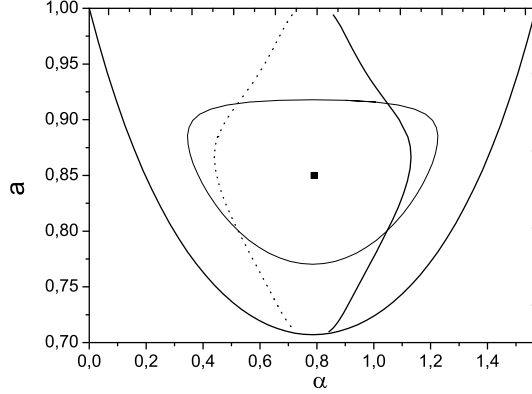


Figure 4: The trajectory of Riemann surface evolution for  $a_0 = 0.8$ ,  $\alpha_0 = \pi/3$  and  $C = 0$ . Bold line corresponds to  $\varepsilon = 1$ , dashed line is for  $\varepsilon = -1$ . Closed curve is the isoperimetric orbit determined by initial data.

$$J_b = \sqrt{\check{J}^2 \sin^2 \check{\Phi} + C \cos^2 \check{\Phi}}, \quad \Phi_b = \varepsilon \frac{\pi}{2}. \quad (39)$$

In principal, we are interested here in the form of trajectory in parameter space  $\mathcal{A}$  without specification of evolution parameter or time. It can be found by means of constraint

$$H\{J[P(a, \alpha)], \Phi[a, P(a, \alpha)], C\} = E, \quad (a, \alpha) \in \mathcal{A}, \quad (40)$$

where constant  $E$  is the value of Hamiltonian function  $H$  for initial data.

Although Eq. (40) says at first sight that the points of trajectory are simply determined from an abstract equation  $h(a, \alpha, C, E) = 0$  with additional constants  $C$  and  $E$ , we would like to emphasize its structure reflecting the chain of our buildings.

In terms of  $a$  and  $\alpha$  the trajectory looks like in Fig. 4, where influence of  $C$  is neglected because of quantum nature assumed. It is interesting to note that the system geometry does not tend to the regular octagon having a maximal information entropy. At the values  $a_{\min} = 1/\sqrt{2}$  and  $a_{\max} = 1$  corresponding to the infinite past and the infinite future time,

angle  $\alpha$  of hyperbolic octagon reaches the same value  $\pi/4$ . Thus, geometry with  $\alpha = \pi/4$  is exceptional in our model. We would like also to note that there are two configurations for  $\varepsilon = \pm 1$  (left and right with respect to  $\tilde{\alpha} = 0$ ), which are not mixed during whole evolution generated by the pure boost. The chosen preference is preserved from the origin to the end. However, we have already seen that there is the diffeomorphism generated by conservation condition of hyperbolic octagon perimeter, which allows us the transition between “phases” with  $\varepsilon = \pm 1$ . Combining it with an action of the boost, it is possible to construct a novel scenario of classical geometrodynamics, even in the case when the Riemannian metric of the parameter (or moduli) space is undetermined.

Coming back to quantization problem, the generator  $J_0$  and its spectrum describes WP-area. For this reason its eigenvalues should be discrete and positive. We choose the irreducible representation (46) with standard basis diagonalizing the Casimir and  $J_0$  and with minimal positive spin  $j = 1/2$ . It leads straightforwardly to the spectrum (in the Planck units):

$$A_{\text{WP}} = 4\pi \left( n + \frac{1}{2} \right), \quad n \in \mathbb{N}. \quad (41)$$

In classical picture,  $A_{\text{WP}} = 0$  for  $P_{\text{reg}}$ , while  $A_{\text{WP}}$  is always non-vanishing at quantum level for the system with a given topology. It means that the regular octagonal configuration is not achieved because of quantum effect.

Although the basis of  $\mathfrak{su}(1, 1)$  is enough to quantize  $A_{\text{WP}}$  as a global characteristic of  $\mathcal{A}$ , it looks insufficient to apply for finding the spectra of other geometric observables. Further investigations are still needed.

## 5 Conclusions

Here, we pay great attention to the structure of admissible parameters space  $\mathcal{A}$  determining the geometry of the Riemann surface in genus two with an order four automorphism. First, using the Weil–Petersson (WP) geometry, two-dimensional space  $\mathcal{A}$  is equipped with the fundamental symplectic two-form. Further, we perform the dense covering of  $\mathcal{A}$  by the orbits generated by the isoperimetric constraint which is imposed on fundamental

domain of the surface. It is argued that an existence of these orbits is due to the Riemann surface definition. The canonically conjugate action–angle variables for isoperimetric orbits are found by identifying WP-area of domain bounded by the orbit with the action variable. As the result, we take on a possibility to construct relativistic model in terms of special invariants without knowing the Riemannian metric of  $\mathcal{A}$ .

To build the physically meaningful model, we extend the set of global canonical variables up to generators of  $\mathfrak{su}(1,1)$  algebra. This trick leads to appearance of the Casimir playing a role of additional parameter. We can only assume that its value should be minimal and non-zero. However, it leads after quantization to non-vanishing discrete spectrum of WP-area in a contrast with initial theory where WP-area becomes zero for the Riemann surface associated with the regular hyperbolic octagon. In any case, we should remember that WP-area determines the surface geometry up to diffeomorphism in classical theory. We may need to use the infinite Witt and Virasoro algebras instead of  $\mathfrak{su}(1,1)$  in order to describe the system spectrum.

We also consider relativistic kinematics or free geometrodynamics of the Riemann surface, generated by the Lorentz boost which acts on the constructed generators of  $\mathfrak{su}(1,1)$  algebra. The time dependence of global variables leads to “big bounce” scenario and is similar for quantities of different origin. However, solving equations with respect to the surface parameters, we have obtained the trajectory (independent on time definition) in space  $\mathcal{A}$ . In this picture, the system does not tend to reach the regular octagon configuration corresponding to the maximal information entropy [6] and preserves some kind of  $\mathbb{Z}_2$  symmetry related to admissibility range of angle variable during whole evolution.

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## A Weil–Petersson Area Estimation

To evaluate the integral (20) analytically, we use the asymptotic expansion:

$$\operatorname{arccosh} z = \ln(2z) - \frac{1}{2 \cdot 2z^2} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 4z^4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6z^4} - \dots \quad (42)$$

Limiting ourselves by accounting for logarithmic term only, we obtain

$$A_{\text{WP}}^{(0)}(P) = F(T, \sqrt{2x_+(T) - 1}) - F(T, \sqrt{2x_-(T) - 1}), \quad (43)$$

where

$$\begin{aligned} F(T, \xi) = & 16 \ln \frac{2}{\sqrt{1 - T^2}} \operatorname{arctanh} \xi + 8 \ln \xi \ln(1 + \xi) + 8 \operatorname{dilog} \xi \\ & + 8 \operatorname{dilog}(1 + \xi) + 4 \sum_{\epsilon = \pm 1} \left\{ \operatorname{dilog} \frac{T + \epsilon \xi}{T - \epsilon} - \operatorname{dilog} \frac{T + \epsilon \xi}{T + \epsilon} \right. \\ & + \operatorname{dilog} \left[ \frac{1 - \xi}{2} + \epsilon i \frac{1 + \xi}{2} \right] - \operatorname{dilog} \left[ \frac{1 + \xi}{2} + \epsilon i \frac{1 - \xi}{2} \right] \\ & \left. + \operatorname{dilog} \frac{\sqrt{2T^2 - 1} + \epsilon \xi}{\sqrt{2T^2 - 1} + \epsilon} - \operatorname{dilog} \frac{\sqrt{2T^2 - 1} + \epsilon \xi}{\sqrt{2T^2 - 1} - \epsilon} \right\}. \end{aligned} \quad (44)$$

Here, the dilogarithm function is defined by the following series:

$$\operatorname{dilog} z = \sum_{p=1}^{\infty} \frac{(1 - z)^p}{p^2}. \quad (45)$$

## B Basis of SU(1,1)

We use the usual basis of SU(1,1) diagonalizing both the Casimir and  $J_0$ . The action of  $\mathfrak{su}(1,1)$  generators on this orthonormal basis is

$$\begin{aligned} C|j, m\rangle &= j(j - 1)|j, m\rangle, & J_0|j, m\rangle &= m|j, m\rangle, \\ J_+|j, m\rangle &= \sqrt{(m - j + 1)(m + j)}|j, m + 1\rangle, \\ J_-|j, m\rangle &= \sqrt{(m - j)(m + j - 1)}|j, m - 1\rangle. \end{aligned} \quad (46)$$

There are two types of (discrete) representations: the positive series with  $1/2 \leq j \leq m = j + \mathbb{N}$ ; and the negative one with  $-1/2 \geq j \geq m = j - \mathbb{N}$ . Here we restrict our consideration by the irreducible representation of positive spin  $j$ .



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